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Elastic fields of interacting elliptic inhomogeneities

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Abstract

This paper presents a method for the calculation of two-dimensional elastic fields in a solid containing any number of inhomogeneities under arbitrary far field loadings. The method called 'pseudo-dislocations method', is illustrated for the solution of interacting elliptic inhomogeneities. It reduces the interacting inhomogeneities problem to a set of linear algebraic equations. Numerical results are presented for a variety of elliptic inhomogeneity arrangements, including the special cases of elliptic holes, cracks and circular inhomogeneities. All these complicated problems can be solved with high accuracy and efficiency.

1. Introduction

In mesoscopic scale, most engineering materials are not homogeneous, but heterogeneous. The deliberately introduced or unwanted inhomogeneities (including defects) in heterogeneous materials, may drastically influence their mechanical behavior. One of the basic problems in solid mechanics is to determine the elastic fields of these heterogeneous materials induced by inhomogeneities. For one single simple geometric inhomogeneity, much work has been done, also for several interacting inhomogeneities, see e.g., Mura (1982) for an account of one inhomogeneity, Atkinson (1972) and Erdogan et al. (1974) for the interaction between one circular inclusion and one crack, Chen and Acrivos (1978) for two equal size spherical inhomogeneities, Mosthoridis and Mura (1975) solved two ellipsoidal inhomogeneities. For many interacting inhomogeneities, commonly used approaches are based on the single inhomogeneity fields, and approximately take account for the interaction among inhomogeneities by some kind of 'effective medium methods', see reviews by Christensen (1990) and Nemat-Nasser and Hori (1993). These approximate methods can only provide an estimation on the overall material response for low inhomogeneity concentrations.

Local mechanical behaviors of heterogeneous materials are sensitive to interacting effects among

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inhomogeneities. To obtain a more realistic description of the behavior, it is appropriate to take into account location and geometric parameters of numerous interacting inhomogeneities. For the difficulty it involved, the problem cannot be solved analytically except in special cases (regular array, simple geometry). In recent years, based on superposition technique, some approximate methods have been proposed to treat the interacting problem of multiple cracks and voids, and the analysis has become an area of research interest (such as, Gross, 1982; Chen, 1984; Kachanov, 1985, 1987, 1992; Kachanov and Laures, 1989; Horii and Nemat-Nasser, 1985; Benvenist et al., 1989; Ju and Tseng, 1992; Hu et al., 1993; Chandra et al., 1995; Huang et al., 1996; Fond and Berthaud, 1995; Han and Wang, 1996). On the other hand, the study of numerous interacting inhomogeneities other than cracks and holes, for its difficulty, is relatively few. A successive approximation method is given by Yu and Sendeckyj (1974) for multiple circular inhomogeneities problems. Nemet-Nasser et al. (1982) studied periodically distributed inhomogeneities problems. Rodin and Hwang (1991) and Rodin (1993) extended the average pseudo-tractions method of Kachanov (1985, 1987) for cracks to the realm of equivalent inclusion method for spherical inhomogeneities.

This paper presents a general technique for the solution of the plane elastic field in an infinite media containing any number of inhomogeneities under far field loadings. This method is as accurate and simple as the widely used pseudo-tractions method for interacting defects, but more general, for defects can be considered as weak inhomogeneities. For complex potential analysis provide an effective tool for two-dimension problems, the method presented here is limited to a plane elastic one and for convenience, we outline the method of pseudo-dislocations for interacting elliptic inhomogeneities in an infinite elastic plane. The analysis is developed for multiple elliptic inhomogeneities of any geometry, orientations, location and elastic moduli. Numerical results are given for some examples of inhomogeneity arrangements.

2. Basic equations and formulations

2.1. *General formula for displacement misfit*

We first give the basic equations and formulations for a single inhomogeneity in an infinite media, where there exists displacement misfit (or called 'dislocations') along the inhomogeneity–matrix interface. The analysis here is limited to plane elastic problem.

Let a closed curve C cut the whole z-plane into two parts S_0 and S_j (see Fig. 1(a)). The elastic moduli in the subdomain S_j (inhomogeneity) are different from those in S_0 (matrix). The elastic moduli in inhomogeneity and matrix are denoted by subscripts j and 0, respectively.

Two-dimensional elastic fields can be described by two complex potentials (Muskhelishvili, 1953). The resultant force functions (X, Y) and the displacements (u, v) can be expressed as:

$$
\begin{aligned}
-Y+iX &= \phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)} \\
2\mu(u+iv) &= \kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)} \qquad (1)
\end{aligned}
$$

where μ is the shear modulus, v is Poisson's ratio, and $\kappa = 3-4\mu$ for plane strain or $\kappa = (3-v)/(1+v)$ for plane stress.

By means of a conformal mapping

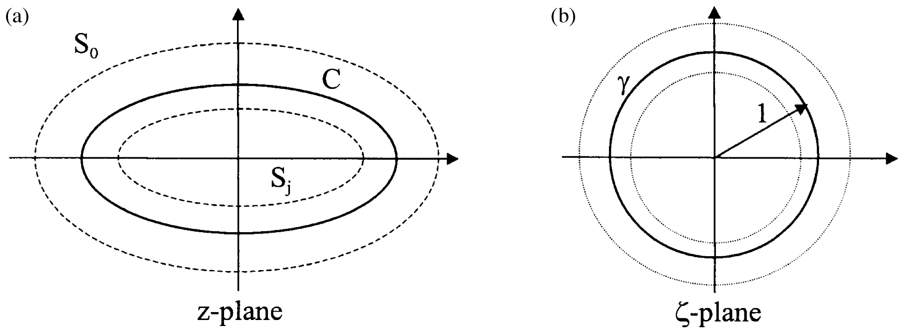

Fig. 1. (a) Elastic plate with an inhomogeneity. (b) Mapping onto the plane deprived of the unit circle.

$$z = \omega(\zeta) \tag{2}$$

the curve C on z-plane is mapped to the unit circumference γ in the plane $\zeta = \rho\,e^{i\theta}$, and the outside of C to the outside of γ (see Fig. 1(b)). The resultant force and the displacement can be expressed as:

$$-Y+iX = \phi(\zeta)+\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)}+\overline{\psi(\zeta)}$$

$$2\mu(u+iv) = \kappa\phi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)}-\overline{\psi(\zeta)} \tag{3}$$

where the convection expression $f[\omega(\zeta)] = f(\zeta)$ is used, and will be used throughout this paper.

It is convenient to replace the potential $\psi(\zeta)$ by the function (Stagni, 1982),

$$\eta(\zeta) = \bar{\omega}\left(\frac{1}{\zeta}\right)\Phi(\zeta)+\psi(\zeta) \tag{4}$$

where $\Phi(\zeta) = \phi'(\zeta)/\omega'(\zeta)$.

Then the resultant force and the displacement can be rewritten as:

$$-Y+iX = \phi(\zeta)+\overline{\eta(\zeta)}+\left[\omega(\zeta)-\omega\left(\frac{1}{\bar{\zeta}}\right)\right]\overline{\Phi(\zeta)}$$

$$2\mu(u+iv) = \kappa\phi(\zeta)-\overline{\eta(\zeta)}-\left[\omega(\zeta)-\omega\left(\frac{1}{\bar{\zeta}}\right)\right]\overline{\Phi(\zeta)} \tag{5}$$

We suppose there exists displacement mismatch or dislocations along the inhomogeneity interface, on the mapped ζ-plane, it is $f_j(e^{i\theta})$.

The resultant force continuity condition and the displacement mismatch condition on interface γ can be written as follows:

$$\phi_j(\sigma)+\overline{\eta_j(\sigma)} = \phi_j^{\mathrm{I}}(\sigma)+\overline{\eta_j^{\mathrm{I}}(\sigma)}, \quad \lambda_j[\kappa_0\phi_j(\sigma)-\overline{\eta_j(\sigma)}]-[\kappa_j\phi_j^{\mathrm{I}}(\sigma)-\overline{\eta_j^{\mathrm{I}}(\sigma)}] = 2\mu_j f_j(\sigma) \tag{6}$$

where $\sigma = e^{i\theta}$, $\lambda_j = \mu_j/\mu_0$, the complex potentials are $\phi_j(z)$ and $\eta_j(z)$ in the matrix, and $\phi_j^{\rm I}(z)$ and $\eta_j^{\rm I}(z)$ in the inhomogeneity.

Since no singularities are assumed on the inhomogeneity interface, it always exists as an annular region around $\rho = 1$ in the mapped plane and a corresponding ring in the z-plane (see Fig. 1), such that the complex functions appearing in eqn (6) are holomorphic in the annular ring. Therefore, the complex functions can be expanded into Laurent series in the region,

$$\phi_j(\zeta) = \sum_{-\infty}^{\infty} c_n\zeta^n, \quad \eta_j(\zeta) = \sum_{-\infty}^{\infty} d_n\zeta^n$$

$$\phi_j^{\rm I}(\zeta) = \sum_{-\infty}^{\infty} a_n\zeta^n, \quad \eta_j^{\rm I}(\zeta) = \sum_{-\infty}^{\infty} b_n\zeta^n \tag{7}$$

We express the dislocations function $f_j(\sigma)$ on γ as:

$$2\mu_j f_j(\sigma) = (\lambda_j\kappa_0 + 1)\sum_{-\infty}^{\infty} A_n\sigma^n \tag{8}$$

and substitute (7) and (8) into (6). Then comparing the coefficients of σ^n, we obtain:

$$c_n + \bar{d}_{-n} = a_n + \bar{b}_{-n},$$
$$\lambda_j(\kappa_0 c_n - \bar{d}_{-n}) - (\kappa_j a_n - \bar{b}_{-n}) = (\lambda_j\kappa_0 + 1)A_n, \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{9}$$

Besides these equations derived from the inhomogeneity interface conditions for the unknown coefficients a_n, b_n, c_n, d_n, A_n and a_{-n}, b_{-n}, d_{-n}, A_{-n}, there exist supplementary equations, one stems for the fact that $\phi_j^{\rm I}(z)$ and $\psi_j^{\rm I}(z)$ are homomorphic inside the inhomogeneity, another from the outside boundary condition.

In the following step, we treat in detail the case of an elliptic inhomogeneity in unbounded matrix.

2.2. *Elliptic inhomogeneity in an infinite plane*

Let L be an ellipse with semi-axes $a = R(1+m)$ and $b = R(1-m)$ on the z-plane (see Fig. 2(a)). It can be mapped to a unit circle on the ζ-plane. The mapping function (2) take the form:

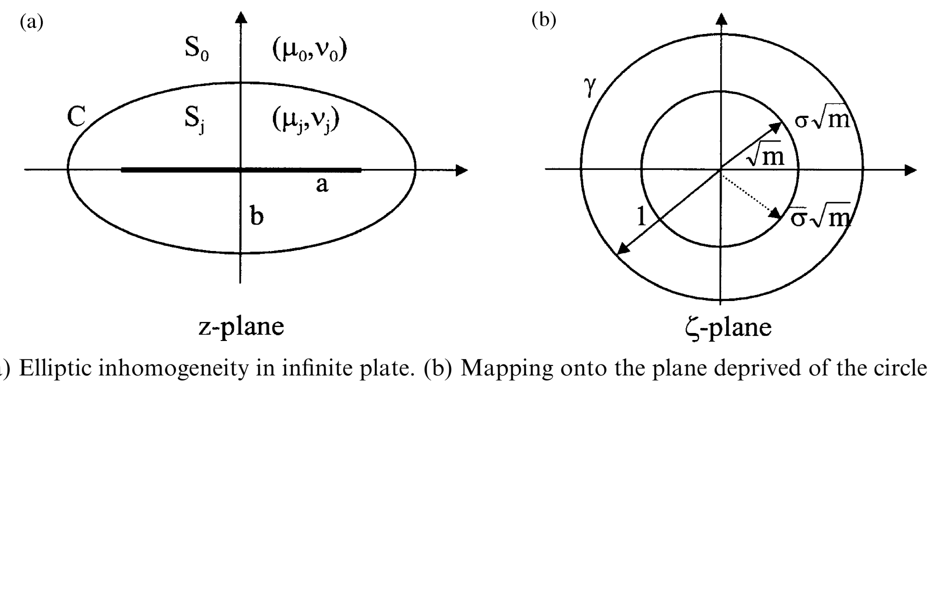

Fig. 2. (a) Elliptic inhomogeneity in infinite plate. (b) Mapping onto the plane deprived of the circle $\rho = \sqrt{m}$.

$$z = \omega(\zeta) = R(\zeta + m/\zeta), \quad R > 0, \quad 0 \leqslant m \leqslant 1 \tag{10}$$

then

$$\eta(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right)\Phi'(\zeta) + \psi(\zeta) = \frac{\zeta(1+m\zeta^2)}{\zeta^2 - m}\phi'(\zeta) + \psi(\zeta) \tag{11}$$

For an infinite matrix under no remote loading, the n-positive coefficients in the Laurent's expansions of $\phi_j(\zeta)$ and $\psi_j(\zeta)$ are vanishing. The same is true of $\eta_j(\zeta)$ (the leading term of $\phi_j'(\zeta)$ being propositional to ζ^{-2}). Hence:

$$c_n = d_n = 0, \quad (n = 1, 2, 3, \ldots) \tag{12}$$

and eqn (9) may be written as:

$$\begin{aligned} \bar{d}_{-n} &= a_n + \bar{b}_{-n}, \\ \lambda_j(-\bar{d}_{-n}) - (\kappa_j a_n - \bar{b}_{-n}) &= (\lambda_j \kappa_0 + 1)A_n, \quad (n = 1, 2, 3, \ldots) \end{aligned} \tag{13}$$

and

$$\begin{aligned} c_{-n} &= a_{-n} + \bar{b}_n, \\ \lambda_j(\kappa_0 c_{-n}) - (\kappa_j a_{-n} - \bar{b}_n) &= (\lambda_j \kappa_0 + 1)A_{-n}, \quad (n = 1, 2, 3, \ldots) \end{aligned} \tag{14}$$

where the zero-order coefficients, which have no effect on the stress field, have been ignored. By suitable linear combinations, eqns (13) and (14) can be expressed further as:

$$\begin{aligned} \bar{d}_{-n} &= a_n + \bar{b}_{-n}, \\ A_n &= (\alpha_j - \beta_j)a_n + (1 - \beta_j)\bar{b}_{-n} \end{aligned} \tag{15}$$

and

$$\begin{aligned} c_{-n} &= a_{-n} + \bar{b}_n, \\ A_{-n} &= \alpha_j a_{-n} + \bar{b}_n \end{aligned} \tag{16}$$

where

$$\alpha_j = \frac{\lambda_j \kappa_0 - \kappa_j}{\lambda_j \kappa_0 + 1}, \quad \beta_j = \frac{\lambda_j(\kappa_0 + 1)}{\lambda_j \kappa_0 + 1} \tag{17}$$

It is noted that the coefficients and so the complex potentials depend only on two elastic parameters of Dundurs (1967).

The mapping function (10) actually maps the whole z-plane, cut alone the segment connecting the ellipse's foci, into the ζ-plane deprived of the circle of radius $\sqrt{m}$ (see Fig. 2(b)). Since $\phi_j^{\mathrm{I}}(z)$ and $\psi_j^{\mathrm{I}}(z)$ are holomorphic inside the elliptic L, they can be expanded into Taylor series. Since $\phi_j^{\mathrm{I}}(\zeta)$ and $\psi_j^{\mathrm{I}}(\zeta)$ are holomorphic on the annular region $\sqrt{m} \leqslant \rho \leqslant 1$, they can be expanded into Laurent series. Then from (11), $\eta_j^{\mathrm{I}}(\zeta)$ can be expressed as Laurent series on the annular ring. The two point $\sigma\sqrt{m}$ and $\bar{\sigma}\sqrt{m} = \sqrt{m}/\sigma$ on the circle $\rho\sqrt{m}\,e^{i\theta}$, correspond to the same point in the z-plane. If a function $f(z)$ is holomorphic inside L, then we have

$$f(\sigma\sqrt{m}) = f(\sqrt{m}/\sigma) \tag{18}$$

Hence, $\phi_j^{\mathrm{I}}(\zeta)$ and $\psi_j^{\mathrm{I}}(\zeta)$ must satisfy (18), it is easily deduced from (7) and (18):

$$a_{-n} = m^n a_n, \quad b_{-n} = m^n b_n + q_n a_n \tag{19}$$

where $q_n = (1-m^2)nm^{n-1}$.

From (15), (16) and (19) the unknown c_{-n}, d_{-n}, A_n, and A_{-n} can also be expressed by a_n and b_n ($n \geqslant$) as:

$$\begin{aligned} c_{-n} &= m^n a_n + \bar{b}_n \\ \bar{d}_{-n} &= a_n + q_n \bar{a}_n + m^n \bar{b}_n \\ A_n &= (\alpha_j - \beta_j) a_n + (1-\beta_j)(q_n \bar{a}_n + m^n \bar{b}_n) \\ A_{-n} &= \alpha_j m^n a_n + \bar{b}_n \end{aligned} \tag{20}$$

So, there are two sets of independent coefficients (a_n, b_n) for the elastic field perturbed by the dislocations along the inhomogeneity interface. They should be determined by the far field loading condition, and the interacting effects with other inhomogeneities (for multiple inhomogeneities).

2.3. Special cases

For an elliptic void, $\lambda_j = 0$, and $\alpha_j = -\kappa_j$, $\beta_j = 0$. And its limiting case for $m = 1$ is a plane sharp crack.

For a rigid elliptical inhomogeneity, $\lambda_j = \infty$, and $\alpha_j = 1$, $\beta_j = 1 + \kappa_0^{-1}$. And its limiting case for $m = 1$ is a rigid line inclusion.

For the special case of a circular inhomogeneity, mapping is no longer needed. Actually, the mapping form (10) for a circular shape takes the form $\zeta = z/R$, which denotes only the position on the z-plane unified by R. In the special simple case, $m = 0$ and $q_1 = 1$, $q_k = 0$ ($k > 1$), the relationship formulae (19) and (20) among coefficients can be simplified by a great deal.

3. Multiple interacting inhomogeneities

3.1. Basic idea—a pseudo-dislocations method

For the interacting effects among inhomogeneities, the problem of a heterogeneous material which containing a large amount of inhomogeneities, is difficult to be solved. Now a general technique—called pseudo-dislocations method, is presented for the calculation of the ealstic field in a homogeneous solid which contains a finite number (M) of nonintersecting and perfectly bonded inhomogeneities. As a usually used method in elastic system (see, for example, Kachanov, 1987 for multiple cracks, and Horrii and Nemat-Nasser, 1985 for multiple holes or cracks), the superposition technique is used first to divide the system into a homogeneous problem and a number (M) of perturbed subproblems. In the homogeneous problem, a homogeneous body (only matrix) witholut any inhomogeneities is subjected to the external applied loading. Each perturbed subproblem concerns only a single inhomogeneity in the matrix with zero external loading, but having an unknown distribution of dislocations (displacement misfit) on the inhomogeneity interface. The unknown distribution of dislocations on each inhomogeneity interface will be determined in such a way that the displacement and traction continuity conditions at all presumed locations of inhomogeneity interfaces are satisfied.

To show the subproblem more clearly, we take any inhomogeneity for consideration. First supposing the inhomogeneity domain is homogeneous with the matrix, calculate the elastic fields in the domain due to external applied loading and the perturbation of other inhomogeneities. Then replacing the matrix in the domain with the inhomogeneity, let the inhomogeneity be under the same stresses states as before. Then the tractions continuity condition on the interface of the inhomogeneity is preserved, but the strains will be different from that of the former, since the elastic moduli of the inhomogeneity are different from those of the matrix. There will exist a pseudo displacement misfit at the inhomogeneity–matrix interface. To eliminate the pseudo displacement misfit, we add an appropriate displacement misfit on the interface, so the displacement continuity condition on the inhomogeneity interface is kept. The displacement misfit is usually called dislocations (see Mura, 1982). Hence we call the method 'pseudo-dislocations method'.

In the following section, the 'pseudo-dislocations method' is illuminated specifically in the solution process for a plane elastic system of an infinite solid containing numerous elliptic inhomogeneities under remote uniform loadings.

3.2. Multiple elliptic inhomogeneities in an infinite plane

Consider an infinitely extended solid plane containing M distinct elliptic inhomogeneities, under far field uniform stresses σ_x^∞, σ_y^∞, σ_{xy}^∞. Besides a global rectangular Cartesian coordinate system xOy, for each inhomogeneity, such as j-ith inhomogeneity, a local Cartesian coordinate system $x_jO_jy_j$ is set up, with origin at the center of the inhomogeneity, and x_j-axis parallel with the major axis of the elliptic inhomogeneity (the axis can be arbitrary for a circular inhomogeneity) and having an orientation angle ϑ_j with respect to the x-axis, see Fig. 3.

Without loss of generality, let us confine attention to the j-th elliptic inhomogeneity with semi-axes $a_j = R_j(1+m_j)$ and $b_j = R_j(1-m_j)$. All quantities associated with the j-th inhomogeneity are denoted by the subscript j. We suppose there exist a distribution of dislocations on the j-th inhomogeneity interface. The z_j-plane is mapped into the ζ_j-plane by the function:

$$z_j = \omega_j(\zeta_j) = R_j(\zeta_j + m_j/\zeta_j) \tag{21}$$

In the mapped plane, we express the complex functions $\phi_j(\zeta_j)$ and $\eta_j(\zeta_j)$ in the infinite matrix region, and $\phi_j^{\mathrm{I}}(\zeta_j)$ and $\eta_j^{\mathrm{I}}(\zeta_j)$ in the inhomogeneity region, as series as:

$$\phi_j(\zeta_j) = \sum_{-\infty}^{-1} c_{n,j}\zeta_j^n, \quad \eta_j(\zeta_j) = \sum_{-\infty}^{-1} d_{n,j}\zeta_j^n$$

$$\phi_j^{\mathrm{I}}(\zeta_j) = \sum_{-\infty}^{\infty} a_{n,j}\zeta_j^n, \quad \eta_j^{\mathrm{I}}(\zeta_j) = \sum_{-\infty}^{\infty} b_{n,j}\zeta_j^n \tag{22}$$

and express the dislocations function $f_j(\sigma_j)$ on the inhomogeneity interface $\sigma_j = \mathrm{e}^{i\theta_j}$ as:

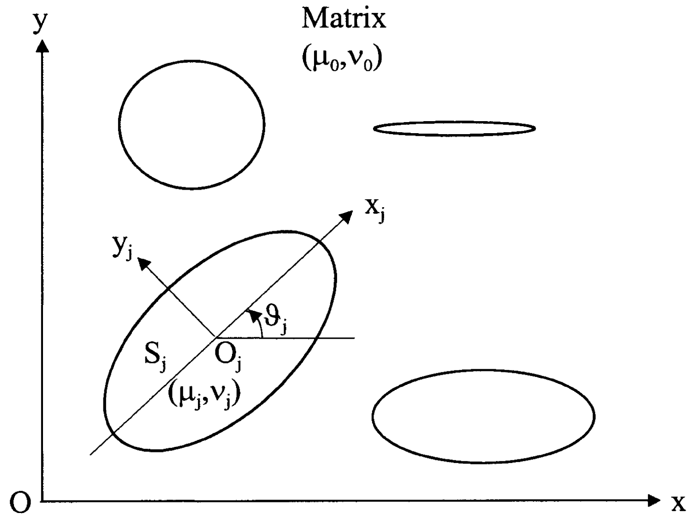

Fig. 3. An infinite plate with multiple elliptic inhomogeneities.

$$f_j(\sigma_j) = \frac{\lambda_j \kappa_0 + 1}{2\mu_j} \sum_{-\infty}^{\infty} A_{n,j} \sigma_j^n \tag{23}$$

Then the complex expansion coefficients satisfy the relation (from Section 2):

$$\begin{aligned} a_{-n,j} &= m_j^n a_{n,j}, \quad b_{-n,j} = m_j^n b_{n,j} + q_{n,j} a_{n,j} \\ c_{-n,j} &= m_j^n a_{n,j} + \bar{b}_{n,j}, \quad \bar{d}_{-n,j} = a_{n,j} + q_{n,j} \bar{a}_{n,j} + m_j^n \bar{b}_{n,j} \\ A_{-n,j} &= \alpha_j m_j^n a_{n,j} + \bar{b}_{n,j}, \quad A_{n,j} = (\alpha_j - \beta_j) a_{n,j} + (1 - \beta_j)(q_{n,j} \bar{a}_{n,j} + m_j^n \bar{b}_{n,j}) \end{aligned} \tag{24}$$

where $q_{n,j} = (1 - m_j^2) n m_j^{n-1}$, and α_j, β_j are two elastic parameters defined in (17).

So there are two sets of independent coefficients $(a_{n,j}, b_{n,j})$ for the perturbation field of the inhomogeneity. They should be determined according to the pseudo-dislocations on the inhomogeneity interface produced by the far field loading and the perturbation fields of all other inhomogeneities (for multiple inhomogeneities system).

The pseudo-dislocations (on the local z_j coordinate system) along the j-th inhomogeneity interface, produced by the applied stresses at infinity, from (1) and noticing coordinate transformation, are:

$$[(u+iv)_j]_{P0} = \frac{1}{2\mu_0}[\kappa_0 \phi_0(z_j) - z_j \overline{\phi_0'(z_j)} - \overline{\psi_0(z_j)}] - \frac{1}{2\mu_j}[\kappa_j \phi_0(z_j) - z_j \overline{\phi_0'(z_j)} - \overline{\psi_0(z_j)}] \tag{25}$$

where

$$z_j = \omega_j(\sigma_j) = R_j(\sigma_j + m_j/\sigma_j), \quad \phi_0(z_j) = \Gamma z_j, \quad \psi_0(z_j) = \Gamma' \mathrm{e}^{2i\vartheta_j} z_j \tag{26}$$

and $\Gamma = (\sigma_x^\infty + \sigma_y^\infty)/4$, $\Gamma' = (\sigma_y^\infty - \sigma_x^\infty)/2 + i\sigma_{xy}^\infty$.

The pseudo-dislocations (on the local z_j coordinate system) along the j-th inhomogeneity interface, produced by the perturbation fields of k-th inhomogeneity, from (4) and noticing coordinate transformation, are:

$$[(u+iv)_j]_{Pk} = \frac{e^{i(\vartheta_k-\vartheta_j)}}{2\mu_0}\left\{\kappa_0\phi_k(\zeta_k)-\overline{\eta_k(\zeta_k)}-\left[\omega_k(\zeta_k)-\omega_k\left(\frac{1}{\bar{\zeta}_k}\right)\right]\overline{\Phi_k(\zeta_k)}\right\} - \frac{e^{i(\vartheta_k-\vartheta_j)}}{2\mu_j}\left\{\kappa_j\phi_k(\zeta_k)-\overline{\eta_k(\zeta_k)}-\left[\omega_k(\zeta_k)-\omega_k\left(\frac{1}{\bar{\zeta}_k}\right)\right]\overline{\Phi_k(\zeta_k)}\right\} \quad (27)$$

where

$$\phi_k(\zeta_k) = \sum_{-\infty}^{-1} c_{n,k}\zeta_k^n, \quad \eta_k(\zeta_k) = \sum_{-\infty}^{-1} d_{n,k}\zeta_k^n \quad (28)$$

and ζ_k is the coordinate of a point in the j-th interface, mapped to the k-th plane, that is

$$z_k\, e^{i\vartheta_k}+O_k = z_j\, e^{i\vartheta_j}+O_j, \quad z_k = \omega_k(\zeta_k) = R_k(\zeta_k+m_k/\zeta_k), \quad z_j = \omega_j(\sigma_j) = R_j(\sigma_j+m_j/\sigma_j) \quad (29)$$

On the j-th inhomogeneity interface, to keep the displacement continuity condition, the added dislocations function $f_j(\sigma_j)$ on the interface, must eliminate the pseudo-dislocations, produced by remote loading and the perturbation fields of all other inhomogeneities. That is:

$$f_j(\sigma_j)+\sum_{\substack{k=1\\k\neq j}}^{M}[(u+iv)_j]Pk = 0, \quad j = 1,\ldots,M \quad (30)$$

Substituting (25), (27) and (23) into (30), the control equations can be expressed further as:

$$\sum_{-\infty}^{\infty} A_{n,j}\sigma_j^n+\sum_{\substack{k=1\\k\neq j}}^{M} e^{i(\vartheta_k-\vartheta_j)}\left\{\alpha_j\phi_k(\sigma_k)+(1-\beta_j)\left\{\overline{\eta_k(\zeta_k)}+\left[\omega_k(\zeta_k)-\omega_k\left(\frac{1}{\bar{\zeta}_k}\right)\right]\overline{\Phi_k(\zeta_k)}\right\}\right\} = -\alpha_j\phi_0(z_j)-(1-\beta_j)[z_j\overline{\phi_0'(z_j)}+\overline{\psi_0(z_j)}] \quad j = 1,\ldots,M \quad (31)$$

Substituting (26), (28) into (31), according to the relation (29), extending ζ_k to the Fourier series of σ_j, we can obtain a system of series equations about σ_j. Comparing the coefficients of σ_j^n, $1 \leqslant |n| \leqslant N$, high order coefficients being ignored, then a system of linear algebraic coefficient equations are obtained. Expressing the coefficients $c_{-n,k}$, $d_{-n,k}$, $A_{-n,k}$ and $A_{n,k}$ in the equations, with $a_{n,k}$ and $b_{n,k}$ according the relationship (24), then we can obtain a system of linear algebraic equations about unknown coefficients $\{a_{n,k}, b_{n,k}\}$ $(1 \leqslant |n| \leqslant N,\ k = 1,2,\ldots,M)$. Solving these equations, all unknown coefficients in M perturbation fields are obtained and the elastic field of the original infinite plane which contains multiple (M) inhomogeneities under far applied stresses, can be obtained by superposition.

The stresses in any point z in the plane can be obtained by the superposition of the homogeneous stress field produced by the far applied loading and the perturbed stress fields of the M subproblem. While the stresses produced by the far applied loadings are,

$$\begin{aligned} \sigma_y+\sigma_x &= 4\,\mathrm{Re}\,\phi_0(z) = 4\Gamma \\ \sigma_y-\sigma_x+2i\sigma_{xy} &= 2[\bar{z}\phi_0''(z)+\phi_0'(z)] = 2\Gamma' \end{aligned} \tag{32}$$

and the perturbed stresses produced by the k-th inhomogeneity are,

$$\begin{aligned} \sigma_y+\sigma_x &= 4\,\mathrm{Re}\,\Phi_k(\zeta_k) \\ \sigma_y-\sigma_x+2i\sigma_{xy} &= \frac{2\,\mathrm{e}^{-2i\vartheta_k}}{\omega_k'(\zeta_k)}[\overline{\omega_k(\zeta_k)}\Phi_k'(\zeta_k)+\psi_k'(\zeta_k)] \\ &= \frac{2\,\mathrm{e}^{-2i\vartheta_k}}{\omega_k'(\zeta_k)}\left\{\left[\overline{\omega_k(\zeta_k)}-\overline{\omega}_k\left(\frac{1}{\zeta_k}\right)\right]\Phi_k'(\zeta_k)+\eta_k'(\zeta_k)-\left[\overline{\omega}_k\left(\frac{1}{\zeta_k}\right)\right]_{\zeta_k}'\Phi_k(\zeta_k)\right\} \end{aligned} \tag{33}$$

where $z = z_k\,\mathrm{e}^{i\vartheta}{}_k+O_k$, $z_k = \omega_k(\zeta_k) = R_k(\zeta_k+m_k/\zeta_k)$. If z is a point in the k-th inhomogeneity region, then the complex potentials ϕ_k and ψ_k (or η_k) in (33) should be replaced by the corresponding potentials ϕ_k^{I} and ψ_k^{I} (or η_k^{I}) of the inhomogeneity.

Especially, the stresses on inhomogeneity–matrix interfaces are usually payed close attention. The stresses along the j-th inhomogeneity interface, are given by the superposition of three parts of elastic fields: the far loading field, the inhomogeneity itself perturbation field and the other inhomogeneities perturbation fields. Here it is convenient to use the local curvilinear coordinate (ρ_j, θ_j), where ρ_j, θ_j is the polar coordinate on the ζ_j-plane. The stresses produced by far loading are:

$$\sigma_{\rho_j}+\sigma_{\theta_j} = 4\,\mathrm{Re}\,\phi_0(z_j) = 4\Gamma, \quad \sigma_{\theta_j}-\sigma_{\rho_j}+2i\sigma_{\rho\theta_j} = 2[\overline{z_j}\phi_0''(z_j)+\psi_0'(z_j)]\,\mathrm{e}^{2i\gamma_j} = 2\Gamma_j'\,\mathrm{e}^{2i\gamma_j} \tag{34}$$

where $\Gamma_j' = \Gamma'\,\mathrm{e}^{2i\vartheta_j}$, and γ_j is the inclination to the x_j-axis of the ρ_j-axis, so $\mathrm{e}^{2i\gamma_j} = \mathrm{e}^{2i\theta_j}\omega_j'(\zeta_j)/\overline{\omega_j'(\zeta_j)}$ and on the j-th interface $\zeta_j = \mathrm{e}^{i\theta_j}$.

The inhomogeneity itself perturbation stress field along the j-th interface on the matrix rim are:

$$\begin{aligned} \sigma_{\rho_j}+\sigma_{\theta_j} &= 2[\Phi_j(\zeta_j)+\overline{\Phi_j(\zeta_j)}] \\ \sigma_{\theta_j}-\sigma_{\rho_j}+2i\sigma_{\rho\theta_j} &= 2\frac{\mathrm{e}^{2i\theta_j}}{\overline{\omega_j'(\zeta_j)}}\left\{\eta_j'(\zeta_j)-\left[\overline{\omega}_j\left(\frac{1}{\zeta_j}\right)\right]_{\zeta_j}'\Phi_j(\zeta_j)\right\} \end{aligned} \tag{35}$$

where $\zeta_j = \mathrm{e}^{i\theta_j} = \sigma_j$. For the stress field along the interface on the inhomogeneity rim, the complex potentials ϕ_j and η_j in the formula should be replaced by the corresponding potentials ϕ_j^{I} and η_j^{I} of the inhomogeneity. The stresses along the j-th interface produced by other inhomogeneities such as k-th are:

$$\begin{aligned} \sigma_{\rho_j}+\sigma_{\theta_j} &= 2[\Phi_k(\zeta_k)+\overline{\Phi_k(\zeta_k)}] \\ \sigma_{\theta_j}-\sigma_{\rho_j}+2i\sigma_{\rho\theta_j} &= 2\frac{\mathrm{e}^{2i(\gamma_j+\vartheta_j-\vartheta_k)}}{\omega_k'(\zeta_k)}[\overline{\omega_k(\zeta_k)}\Phi_k'(\zeta_k)+\psi_k'(\zeta_k)] \\ &= 2\frac{\mathrm{e}^{2i(\gamma_j+\vartheta_j-\vartheta_k)}}{\omega_k'(\zeta_k)}\left\{\left[\overline{\omega_k(\zeta_k)}-\overline{\omega}_k\left(\frac{1}{\zeta_k}\right)\right]\Phi_k'(\zeta_k)+\eta_k'(\zeta_k)-\left[\overline{\omega}_k\left(\frac{1}{\zeta_k}\right)\right]_{\zeta_k}'\Phi_k(\zeta_k)\right\} \end{aligned} \tag{36}$$

where ζ_k is the position coordinate of a point z_j (σ_j on the ζ_j-plane) in the j-th interface, on the mapped ζ_k-plane, and the relationship between ζ_k and z_j is given by (29).

For cracks, the special case of inhomogeneity, their stress intensity factors (SIFs) are especially interested. The stress intensity factors at crack tips of the j-th crack can be calculated by the formula,

$$k_{1j}^{\pm} - ik_{2j}^{\pm} = 2\sqrt{2}\lim_{z_j \to \pm a_j}[\sqrt{z \mp a_j}\phi_j'(z_j)] \tag{37}$$

where the quantities with upper and lower signs refer to the right- and left-hand crack tips, respectively.

They can be expressed further as,

$$\begin{aligned} k_{1j}^{\pm} - ik_{2j}^{\pm} &= \frac{2}{\sqrt{a_j}}\lim_{\zeta_j \to \pm 1}\phi_j'(\zeta_j) \\ &= -\frac{2}{\sqrt{a_j}}\sum_{1}^{\infty}(\pm 1)^{-n-1}nc_{-n,j} \end{aligned} \tag{38}$$

4. Numerical results and discussion

Application of the proposed pseudo-locations method to several example problems involving interactions of elliptic inhomogeneities and holes, cracks is presented in this section. The method can handle arbitrary distributions of elliptic inhomogeneities, including holes and cracks. In this section, the numerical results from the proposed method are first verified against existing solutions in the literature. Problems involving general systems of inhomogeneities are addressed next.

4.1. One elliptic inhomogeneity

As an example, consider the case of an infinite plane containing an elliptic inhomogeneity subjected to general remore loading conditions. For having only one inhomogeneity in the problem, let the global coordinate be identical with the local one. From (31), the control equation here is:

$$\begin{aligned} \sum_{-\infty}^{\infty} A_n\sigma^n &= -\alpha_j\phi_0(z) - (1-\beta_j)[z\overline{\phi_0'(z)} + \overline{\psi_0(z)}] \\ &= -[(\alpha_j + 1 - \beta_j)\Gamma m(1-\beta_j)\overline{\Gamma'}]R\sigma^{-1} - [(\alpha_j + 1 - \beta_j)\Gamma + (1-\beta_j)\overline{\Gamma'}m]R\sigma \end{aligned} \tag{39}$$

Comparing the coefficients of σ^n, we obtain:

$$\begin{aligned} &A_{-1} = -[(\alpha_j + 1 - \beta_j)\Gamma m + (1-\beta_j)\overline{\Gamma'}]R, \quad A_1 = -[(\alpha_j + 1 - \beta_j)\Gamma + (1-\beta_j)\overline{\Gamma'}m]R; \\ &A_{-n} = A_n = 0 \;\; (n > 1) \end{aligned} \tag{40}$$

and from the relation (19) and (20), all coefficients can be obtained. Really, besides the ± 1 terms, all other terms are zero.

The potentials of the perturbation field in the inhomogeneity are:

$$\phi_j^I(\zeta) = a_1(\zeta + m/\zeta), \quad \psi_j^I(\zeta) = (b_1 - ma_1)(\zeta + m/\zeta) \tag{41}$$

or

$$\phi_j^I(z) = a_1 z/R, \quad \psi_j^I(z) = (b_1 - ma_1)z/R \tag{42}$$

From it, we can see that the stress field in the inhomogeneity is uniform, as be known well.

The potentials of the perturbation field in the matrix are:

$$\phi_j(\zeta) = (ma_1 + \bar{b}_1)/\zeta, \quad \psi_j(\zeta) = [(1-m^2)a_1 + \bar{a}_1 + mb_1]/\zeta + \frac{m\zeta^2+1}{\zeta(\zeta^2-m)}(ma_1 + \bar{b}_1) \tag{43}$$

These potentials plus the homogeneous ones $\phi_0(z) = \Gamma z$, $\phi_0(z) = \Gamma' z$, will be the total and exact ones, see Hardiman (1954).

For the special case of a circle inhomogeneity, $m = 0$, and $a_1 = -[(\alpha_j + 1 - \beta_j)/(\alpha_j + 1 - 2\beta_j)]\Gamma R$, $b_1 = (\beta_j - 1)\Gamma' R$. The potentials of the perturbation field are:

$$\begin{aligned}
&\phi_j^{\mathrm{I}}(z) = a_1 = -\frac{\alpha_j + 1 - \beta_j}{\alpha_j + 1 - 2\beta_j}\Gamma z, \quad \psi_j^{\mathrm{I}}(z) = (\beta_j - 1)\Gamma' z \\
&\qquad \phi_j(\zeta) = (\beta_j - 1)\overline{\Gamma'}R^2/z, \\
&\qquad \psi_j(\zeta) = -2\frac{\alpha_j + 1 - \beta_j}{\alpha_j + 1 - 2\beta_j}\Gamma R^2/z + (\beta_j - 1)\overline{\Gamma'}R^4/z^3
\end{aligned} \tag{44}$$

For the special case of an elliptic hole, $\lambda = 0$, $\alpha_j = -\kappa_j$, $\beta_j = 0$, and $a_1 = -\Gamma R$, $b_1 = -(\Gamma R + \Gamma' R)$. The potentials of the perturbation field are:

$$\begin{aligned}
&\phi_j^{\mathrm{I}}(z) = -\Gamma z, \quad \psi_j^{\mathrm{I}}(z) = -\Gamma' z \\
&\phi_j(\zeta) = -(2\Gamma m + \overline{\Gamma'})R/\zeta, \\
&\psi_j(\zeta) = -\frac{2\Gamma R + \Gamma' m R}{\zeta} - \frac{m\zeta^2+1}{\zeta(\zeta^2-m)}(2\Gamma R m + \overline{\Gamma'}R)
\end{aligned} \tag{45}$$

For the special case of a crack, $m = 0$, $R = a/2$, $\lambda = 0$, $\alpha_j = -\kappa_j$, $\beta_j = 0$, and $a_1 = -\Gamma R$, $b_1 = -(\Gamma + \Gamma')R$. The potentials of the perturbation field in the matrix are:

$$\begin{aligned}
&\phi_j(\zeta) = -(2\Gamma + \overline{\Gamma'})R/\zeta, \\
&\psi_j(\zeta) = -\frac{2\Gamma + \Gamma'}{\zeta}R - \frac{\zeta^2+1}{\zeta(\zeta^2-1)}(2\Gamma + \overline{\Gamma'})R
\end{aligned} \tag{46}$$

and it is more usual to express ζ with z, where $\zeta = (z/a) - \sqrt{(z^2/a^2) - 1}$.

The potentials of the perturbation fields (42)–(44) plus the homogeneous ones $\phi_0(z) = \Gamma z$, $\psi_0(z) = \Gamma' z$, will be the total and exact ones, respectively, see Muskhelishvili (1953). It can be seen that the total potentials in holes are zero.

4.2. Multiple inhomogeneities

As a special case of two inhomogeneities, two collinear cracks with equal length, $2a$, separated by distance d, subject to a remote tension, is considered, see Fig. 4. The normalized stress intensity factors (SIFs) at the two tips of the crack are given in Table 1. The results converge to the exact solution of Erdogan (1962), as the number of series terms N is increased. The results do not change for N greater than for those indicated in the table. Table 2 shows the calculation convergence of the normalized SIFs at the inner crack tip A of two equal collinear cracks. The two tables give a show to the accuracy and efficiency of the present method. In the following examples, the series terms N used are similar as those terms in this example, which change from few to twenties (or thirties) terms according to the distance between inhomogeneities (the nearer distance, the more terms are needed). For a definite distance, if the calculated result does not change further when more series terms are used, the result is considered to be convergent.

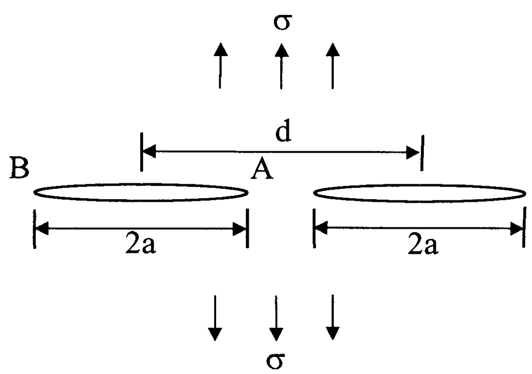

Fig. 4. Two equal collinear cracks under remote tension.

Table 1
The normalized SIFs for two equal collinear cracks

		$k_1/\sigma\sqrt{\pi a}$ at A		$k_1/\sigma\sqrt{\pi a}$ at B	
$2a/d$	N	Present	Exact	Present	Exact
0.10	2	1.00132	1.00132	1.00120	1.00120
0.20	4	1.00566	1.00566	1.00462	1.00462
0.30	4	1.01383	1.01383	1.01017	1.01017
0.40	5	1.02717	1.02717	1.01787	1.01787
0.50	6	1.04796	1.04796	1.02795	1.02795
0.60	7	1.08040	1.08040	1.04094	1.04094
0.70	8	1.13326	1.13326	1.05786	1.05786
0.80	14	1.22894	1.22894	1.08107	1.08107
0.90	17	1.45387	1.45387	1.11741	1.11741

Table 2
The convergence of the normalized SIFs at the inner crack tip A of two equal collinear crack

$2a/d$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$N=2$	1.00132	1.00562	1.01360	1.02633	1.04555	1.07419	1.11783	1.18880	1.32388
$N=4$	—	1.00566	1.01383	1.02716	1.04788	1.08008	1.13200	1.22365	1.42382
$N=5$	—	—	—	1.02717	1.04795	1.08033	1.13291	1.22703	1.43931
$N=6$	—	—	—	—	1.04796	1.08039	1.13316	1.22825	1.44676
$N=7$	—	—	—	—	—	1.08040	1.13324	1.22869	1.45038
$N=8$	—	—	—	—	—	—	1.13326	1.22885	1.45214
$N=14$	—	—	—	—	—	—	—	1.22894	1.45384
$N=17$	—	—	—	—	—	—	—	—	1.45387
$N=20$	1.1.00132	1.00566	1.01383	1.02717	1.04796	1.08040	1.13326	1.22894	1.45387

Figure 5(a) shows a crack aligned along an axis of an elliptic inhomogeneity, subjected to a remote tension. Assuming the inhomogeneity is a circular one with radius $R = 2a_1$, the normalized SIF at the crack near tip A is shown in Fig. 5(b) against the distance from the inhomogeneity t/R, for $\nu_2 = \nu_0 = 0.25$ and various value μ_2/μ_0. From the figure, we can see that the result agrees very well with those obtained by Atkinson (1972). Assuming $a_1 = a_2$, the normalized SIF at the tip A is shown in Fig. 5(c) against $2a_1/d$, for $\nu_2 = \nu_0 = 0.3$, and various value μ_2/μ_0 and b_1/a_2. As the crack approaches the inhomogeneity the SIF increases for the soft inhomogeneity ($\mu_2/\mu_0 < 1$), the SIF decreases for the hard inhomogeneity ($\mu_2/\mu_0 > 1$). Generally, the amplification or retardation effect of the SIF is more notable for bigger b_2/a_2, especially for approaching hard inhomogeneities, but the effect is weakened for very large b_2/a_2 and $2a_1/d$ (for $b_2/a_2 \to \infty$ and $2a_1/d \to 1$, it is hard to obtain convergent and accurate results by this method). It is noted that the perturbation of stress intensity factors due to the inhomogeneity becomes very small when $d/2a_1 > 2$.

Figure 6(a) shows a circular hole aligned along an axis of an elliptic inhomogeneity under a remote tension. We take the case where $R = a_2$ and $\nu_2 = \nu_0 = 0.3$. The hoop stress at the hole near point A is shown in Fig. 6(b) against $2R/d$, for different value μ_2/μ_0 and b_2/a_2. It can be seen that

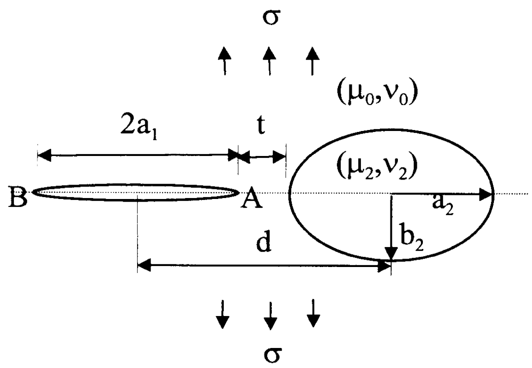

Fig. 5(a). A crack and an elliptic inhomogeneity under remote tension.

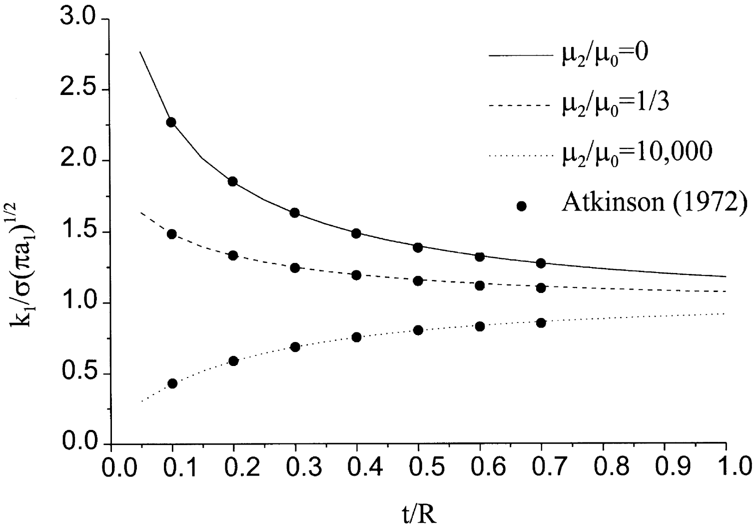

Fig. 5(b). The normalized SIFs at the crack tip A against T/R, with $a_2 = b_2 = R = 2a_1$ and $v_2 = v_0 = 0.25$.

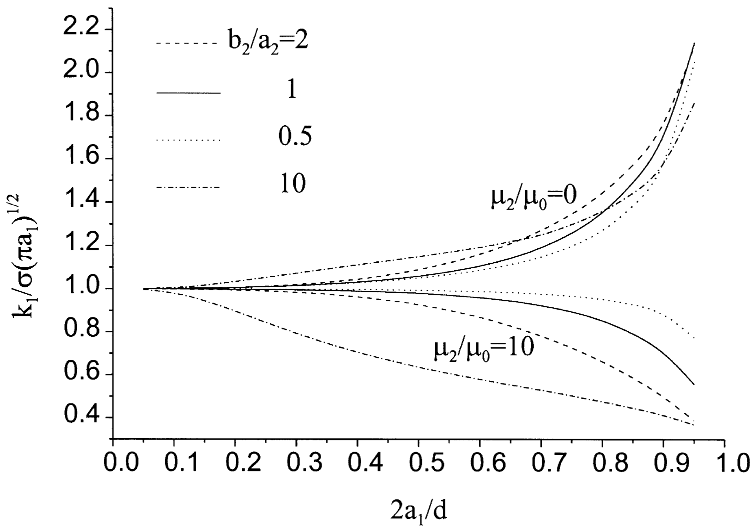

Fig. 5(c). The normalized SIFs at the crack tip A against $2a_1/d$, with $a_2 = a_1$ and $v_2 = v_0 = 0.3$.

the hoop stress at the near point A increases, as the hole approaches the soft inhomogeneity, decreases as it approaches the hard inhomogeneity, and generally the changing magnitude is greater for bigger b_2/a_2. The hoop stress perturbation along the hole due to the interacting effect, becomes very small when $d/2R > 2$. Generally speaking, the maximum hoop stress alone holes may exist at the nearest points. When the spacing between the hole and the inhomogeneity is very small, the maximum hoop stress may not happen at the hole nearest point. This can be seen from Fig. 6(c), which shows the hoop stress along the rims of a circular hole and an oblate hole, with

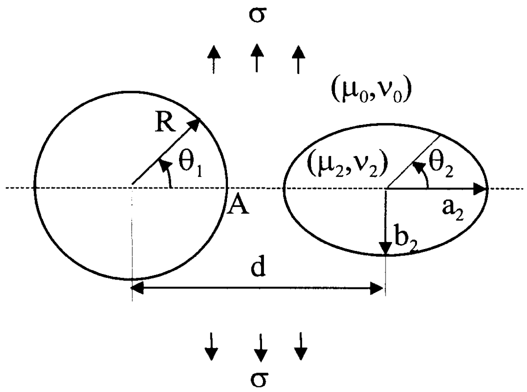

Fig. 6(a). A circular hole and an elliptic inhomogeneity under remote tension.

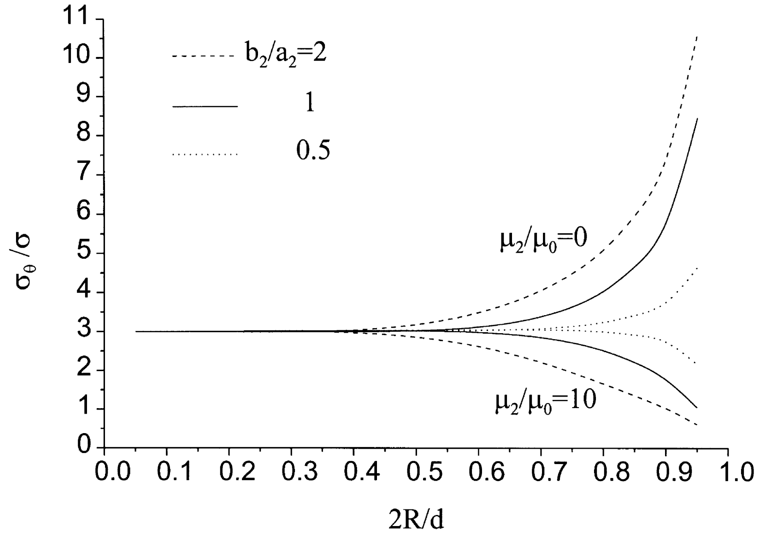

Fig. 6(b). The hoops stress at the near point A of the hole rim against $2R/d$, with $R = a_2$ and $\nu_2 = \nu_0 = 0.3$.

$R = a_2$, $2R/d = 0.9$ and $b_2/a_2 = 0.5$. On the circular hole rim, the maximum hoop stress, happened at about $\theta_1 = 14^\circ$ diverging from the nearest point A.

Figure 7(a) shows a crack surrounded by a square array of identical elliptic inhomogeneities under remote tension. The normalized SIFs of the crack surrounded by holes and hard inhomogeneities are shown in Fig. 7(b). When the crack is small, the crack tips are far from the inhomogeneities, the surrounding holes have retardation effect to the crack. When the crack is large, the crack tips are near to the inhomogeneities, the surrounding holes have amplification effect to the crack. For the crack surrounded by hard inhomogeneities, the interacting effect is opposite with that by surrounding holes.

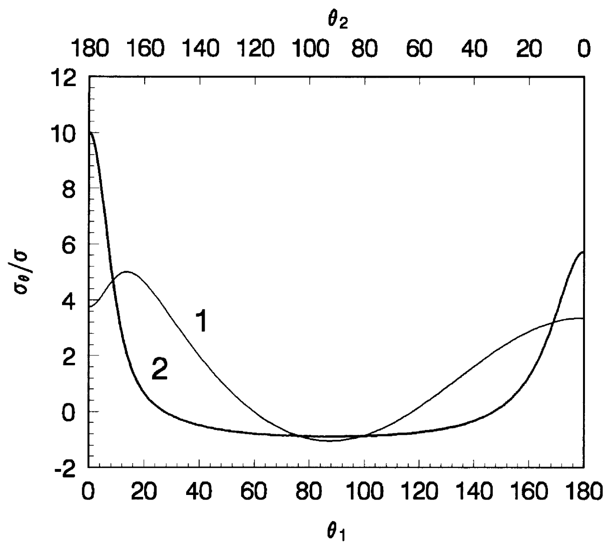

Fig. 6(c). The hoop stress along the circular rim and the elliptic hole rim, with $b_2/a_2 = 0.5$, $R = a_2$ and $2R/d = 0.9$.

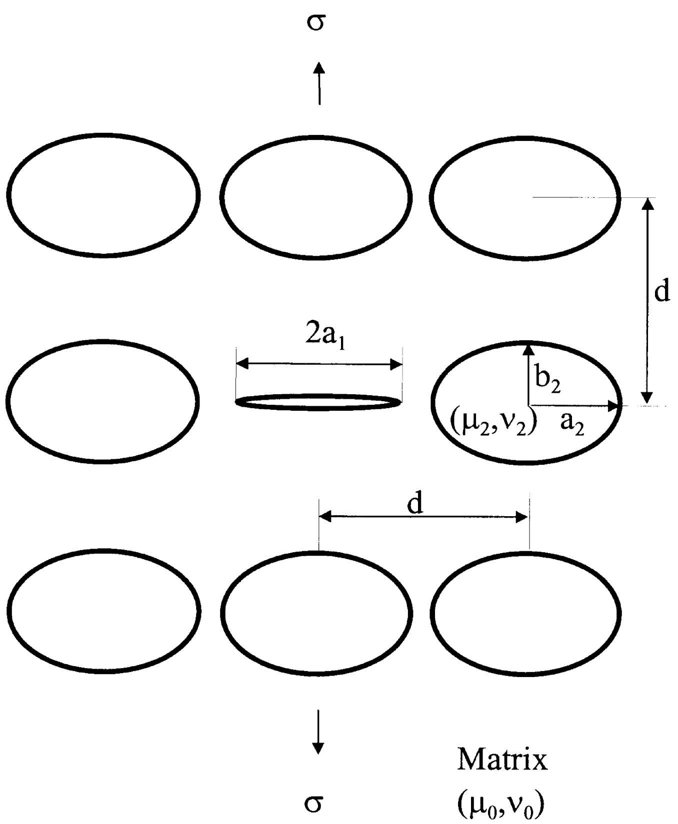

Fig. 7(a). A crack surrounded by a square array of identical elliptic inhomogeneities under remote tension.

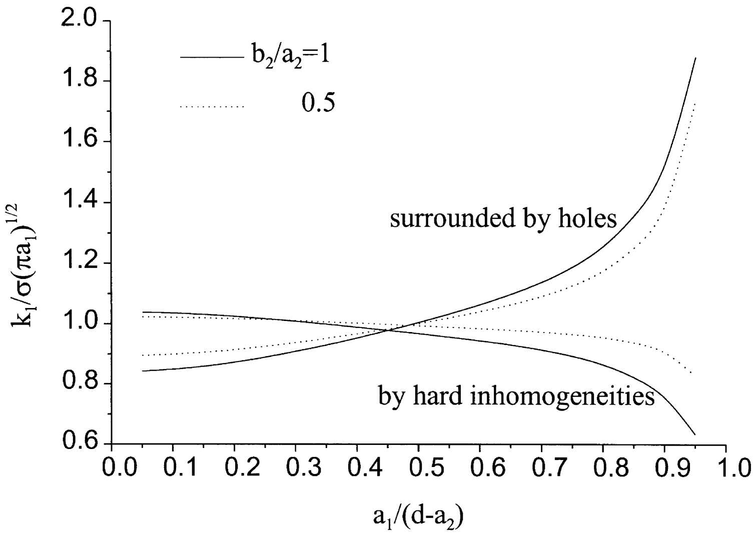

Fig. 7(b). The normalized SIFs of the surrounded crack against $\delta = a_1/(d-a_2)$, with $d/a_2 = 4$, for the surrounding inhomogeneities $\mu_2 = 10\mu_0$ and $v_2 = v_0 = 0.3$.

5. Conclusions

A pseudo-dislocation method has been developed to treat the interacting problem among inhomogeneities including defects. The method is illustrated for interacting elliptic inhomogeneities in the context of two-dimensional elastic mechanics. It is nearly capable of handling general loading conditions and arbitrary orientations, locations and geometric sizes of elliptic inhomogeneities with arbitrary elastic moduli. Expressing the pseudo-dislocations in terms of series, the governing equations is reduced to a system of easily solved algebraic equations. The solution usually converges quickly and can be obtained accurately.

Since the present method has high computative efficiency and accuracy, it may provide an efficient tool to deal with micromechanical problems of heterogeneous solid materials, such as to numerically obtain the effective moduli and directly verify various kinds of effective medium methods, to verify directly some micromechanical models of interacting inhomogeneities, to predict damage process of these materials, etc.

Acknowledgements

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